On the stochastic nonlinear Schrödinger equations at critical regularities.

Stoch PDE: Anal Comp
https://doi.org/10.1007/s40072-019-00163-5

On the stochastic nonlinear Schrödinger equations at critical regularities

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Received: 14 April 2019 / Revised: 20 November 2019

Abstract
We consider the Cauchy problem for the defocusing stochastic nonlinear Schrödinger equations (SNLS) with an additive noise in the mass-critical and energy-critical settings. By adapting the probabilistic perturbation argument employed in the context of the random data Cauchy theory by Bényi et al. (Trans Am Math Soc Ser B 2:1–50, 2015) to the current stochastic PDE setting, we present a concise argument to establish global well-posedness of the mass-critical and energy-critical SNLS.

Keywords Stochastic nonlinear Schrödinger equation · Global well-posedness · Mass-critical · Energy-critical · Perturbation theory

Mathematics Subject Classification 35Q55

1 Introduction

1.1 Stochastic nonlinear Schrödinger equations

We consider the Cauchy problem for the stochastic nonlinear Schrödinger equation (SNLS) with an additive noise:

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$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u + \phi\xi & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u|_{t=0} = u_0, \end{cases} \tag{1}$$

where $\xi(t,x)$ denotes a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and ϕ is a bounded operator on $L^2(\mathbb{R}^d)$. In this paper, we restrict our attention to the defocusing case. Our main goal is to present a concise argument in establishing global well-posedness of (1) in the so-called *mass-critical* and *energy-critical* cases.

Let us first go over the notion of the scaling-critical regularity for the (deterministic) defocusing nonlinear Schrödinger equation (NLS):

$$i\partial_t u + \Delta u = |u|^{p-1}u, \tag{2}$$

namely, (1) with $\phi \equiv 0$. The Eq. (2) is known to enjoy the following dilation symmetry:

$$u(t,x) \longmapsto u^\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u(\lambda^{-2}t, \lambda^{-1}x)$$

for $\lambda > 0$. If u is a solution to (2), then the scaled function u^λ is also a solution to (2) with the rescaled initial data. This dilation symmetry induces the following scaling-critical Sobolev regularity:

$$s_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1}$$

such that the homogeneous $\dot{H}^{s_{\text{crit}}}(\mathbb{R}^d)$-norm is invariant under the dilation symmetry. This critical regularity s_{crit} provides a threshold regularity for well-posedness and ill-posedness of (2). Indeed, when $s \geq \max(s_{\text{crit}}, 0)$, the Cauchy problem (2) is known to be locally well-posed in $H^s(\mathbb{R}^d)$ [6,19,22,36].[1] On the other hand, it is known that NLS (2) is ill-posed in the scaling supercritical regime: $s < s_{\text{crit}}$. See [9,26,28].

Next, we introduce two important critical regularities associated with the following conservation laws for (2):

$$\begin{aligned} \text{Mass: } M(u(t)) &:= \int_{\mathbb{R}^d} |u(t,x)|^2 dx, \\ \text{Energy: } E(u(t)) &:= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx + \frac{d-2}{2d}\int_{\mathbb{R}^d} |u(t,x)|^{\frac{2d}{d-2}} dx. \end{aligned}$$

In view of these conservation laws, we say that the Eq. (2) is

(i) *Mass-critical* when $s_{\text{crit}} = 0$, namely, when $p = 1 + \frac{4}{d}$,
(ii) *Energy-critical* when $s_{\text{crit}} = 1$, namely, when $p = 1 + \frac{4}{d-2}$ and $d \geq 3$.

Over the last two decades, we have seen a significant progress in the global-in-time theory of the defocusing NLS (2) in the mass-critical and energy-critical cases [5,11, 15–17,31,34,37]. In particular, we now know that

[1] When p is not an odd integer, we may need to impose an extra assumption due to the non-smoothness of the nonlinearity.

(i) The defocusing mass-critical NLS (2) with $p = 1 + \frac{4}{d}$ is globally well-posed in $L^2(\mathbb{R}^d)$,
(ii) The defocusing energy-critical NLS (2) with $p = 1 + \frac{4}{d-2}$, $d \geq 3$, is globally well-posed in $\dot{H}^1(\mathbb{R}^d)$.

Moreover, the following space-time bound on a global solution u to (2) holds:

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2k}}(\mathbb{R}\times\mathbb{R}^d)} \leq C(\|u_0\|_{H^k}) < \infty \tag{3}$$

with (i) $k = 0$ in the mass-critical case and (ii) $k = 1$ in the energy-critical case. This bound in particular implies that global-in-time solutions scatter, i.e. they asymptotically behave like linear solutions as $t \to \pm\infty$.

Let us now turn our attention to SNLS (1). We say that u is a solution to (1) if it satisfies the following Duhamel formulation (= mild formulation):

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')|u|^{p-1}u(t')dt' - i\int_0^t S(t-t')\phi\xi(dt'), \tag{4}$$

where $S(t) = e^{it\Delta}$ denotes the linear Schrödinger propagator. The last term on the right-hand side of (4) is called the stochastic convolution, which we denote by Ψ. Fix a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and let W denote the $L^2(\mathbb{R}^d)$-cylindrical Wiener process associated with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$; see (10) below for a precise definition. Then, the stochastic convolution Ψ is defined by

$$\begin{aligned}\Psi(t) &= -i\int_0^t S(t-t')\phi\xi(dt')\\ &:= -i\int_0^t S(t-t')\phi dW(t').\end{aligned} \tag{5}$$

See Sect. 2 for the precise meaning of the definition (5); in particular see (11).

Our main goal is to construct global-in-time dynamics for (4) in the mass-critical and energy-critical cases. This means that we take (i) $p = 1+\frac{4}{d}$ in the mass-critical case and (ii) $p = 1 + \frac{4}{d-2}$ in the energy-critical case. Furthermore, we take the stochastic convolution Ψ in (5) to be at the corresponding critical regularity. Suppose that $\phi \in HS(L^2; H^s)$, namely, ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$. Then, it is known that $\Psi \in C(\mathbb{R}_+; H^s(\mathbb{R}^d))$ almost surely; see [12]. Therefore, we will impose that (i) $\phi \in HS(L^2; L^2)$ in the mass-critical case and (ii) $\phi \in HS(L^2; H^1)$ in the energy-critical case.

Previously, de Bouard and Debussche [14] studied SNLS (1) in the energy-subcritical setting: $s_{\text{crit}} < 1$, assuming that $\phi \in HS(L^2; H^1)$. By using the Strichartz estimates, they showed that the stochastic convolution Ψ almost surely belongs to a right Strichartz space, which allowed them to prove local well-posedness of (1) in $H^1(\mathbb{R}^d)$ with $\phi \in HS(L^2; H^1)$ in the energy-subcritical case: $1 < p < 1+\frac{4}{d-2}$ when $d \geq 3$ and $1 < p < \infty$ when $d = 1, 2$. We point out that when $s \geq \max(s_{\text{crit}}, 0)$, a slight modification of the argument in [14] with the regularity properties of the stochastic convolution (see Lemma 2.2 below) yields local well-posedness[2] of (1) in $H^s(\mathbb{R}^d)$, provided that $\phi \in HS(L^2; H^s)$. See Lemma 2.3 for the statements in the mass-critical and energy-critical cases. We also mention recent papers [8,30] on local well-posedness of (1) with additive noises rougher than the critical regularities, i.e. $\phi \in HS(L^2; H^s)$ with $s < s_{\text{crit}}$.

In the energy-subcritical case, assuming $\phi \in HS(L^2; H^1)$, global well-posedness of (1) in $H^1(\mathbb{R}^d)$ follows from an a priori H^1-bound of solutions to (1) based on the conservation of the energy $E(u)$ for the deterministic NLS and Ito's lemma; see [14]. See also Lemma 2.4. In a recent paper [7], Cheung et al. adapted the I-method [10] to the stochastic PDE setting and established global well-posedness of energy-subcritical SNLS below $H^1(\mathbb{R}^d)$. In the mass-subcritical case, global well-posedness in $L^2(\mathbb{R}^d)$ also follows from an a priori L^2-bound based on the conservation of the mass $M(u)$ for the deterministic NLS and Ito's lemma.

We extend these global well-posedness results to the mass-critical and energy-critical settings.

Theorem 1.1 (i) *(Mass-critical case). Let $d \geq 1$ and $p = 1 + \frac{4}{d}$. Then, given $\phi \in HS(L^2; L^2)$, the defocusing mass-critical SNLS* (1) *is globally well-posed in $L^2(\mathbb{R}^d)$.*

(ii) *(Energy-critical case). Let $3 \leq d \leq 6$ and $p = 1 + \frac{4}{d-2}$. Then, given $\phi \in HS(L^2; H^1)$, the defocusing energy-critical SNLS* (1) *is globally well-posed in $H^1(\mathbb{R}^d)$.*

In the following, we only consider deterministic initial data u_0. This assumption is, however, not essential and we may also take random initial data (measurable with respect to the filtration $\mathcal{F}_0$ at time 0).

In the mass-critical case (and the energy-critical case, respectively), the a priori L^2-bound (and the a priori H^1-bound, respectively) does not suffice for global well-posedness (even in the case of the deterministic NLS (2)). The main idea for proving Theorem 1.1 is to adapt the probabilistic perturbation argument introduced by the authors [4,29] in studying global-in-time behavior of solutions to the defocusing energy-critical cubic NLS with random initial data below the energy space. Namely, by letting $v = u - \Psi$, where Ψ is the stochastic convolution defined in (5), we study the equation satisfied by v:

$$\begin{cases} i\partial_t v + \Delta v = \mathcal{N}(v + \Psi) \\ v|_{t=0} = u_0, \end{cases} \tag{6}$$

where $\mathcal{N}(u) = |u|^{p-1}u$. Write the nonlinearity as

$$\mathcal{N}(v + \Psi) = \mathcal{N}(v) + \big(\mathcal{N}(v + \Psi) - \mathcal{N}(v)\big).$$

Then, the regularity properties of the stochastic convolution (see Lemma 2.2 below) and the fact that their space-time norms can be made small on short time intervals

[2] When p is not an odd integer, an extra assumption such as $p \geq [s] + 1$ may be needed.

allow us to view the second term on the right-hand side as a perturbative term. By invoking the perturbation lemma (Lemmas 3.2, 4.3), we then compare the solution v to (6) with a solution to the deterministic NLS (2) on short time intervals as in [4,29]. See also [24,35] for similar arguments in the deterministic case. In the energy-critical case, we rely on the Lipschitz continuity of $\nabla\mathcal{N}(u)$ in the perturbation argument, which imposes the assumption $d \leq 6$ in Theorem 1.1.

Remark 1.2 We remark that solutions constructed in this paper are adapted to the given filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For example, adaptedness of a solution v to (6) directly follows from the local-in-time construction of the solution via the Picard iteration. Namely, we consider the map Γ defined by

$$\Gamma v(t) := S(t)u_0 - i\int_0^t S(t-t')\mathcal{N}(v+\Psi)(t')dt'.$$

Then, we define the jth Picard iterate P_j by setting

$$\begin{aligned} P_1 &= S(t)u_0, \\ P_{j+1} &= \Gamma P_j = S(t)u_0 - i\int_0^t S(t-t')\mathcal{N}(P_j+\Psi)(t')dt' \end{aligned} \tag{7}$$

for $j \in \mathbb{N}$. Since the stochastic convolution Ψ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, it is easy to see from (7) that P_j is adapted for each $j \in \mathbb{N}$. Furthermore, the local well-posedness of (6) by a contraction mapping principle (see Lemmas 3.1 and 4.1 below) shows that the sequence $\{P_j\}_{j\in\mathbb{N}}$ converges, in appropriate functions spaces, to a limit $v = \lim_{j\to\infty} P_j$, which is a solution to (the mild formulation of) (6). By invoking the closure property of measurability under a limit, we conclude that the solution v to (6) is also adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The same comment applies to Lemma 2.3 below.

Remark 1.3 (i) In the focusing case, i.e. with $-|u|^{p-1}u$ in (1), de Bouard and Debussche [13] proved under appropriate conditions that, starting with any initial data, finite-time blowup occurs with positive probability.

(ii) In the mass-subcritical and energy-critical cases, SNLS with a multiplicative noise has been studied in [1–3]. In recent preprints, Fan and Xu [18] and Zhang [39] proved global well-posedness of SNLS with a multiplicative noise in the mass-critical and energy-critical setting.

2 Preliminary results

In this section, we introduce some notations and go over preliminary results.

Given two separable Hilbert spaces H and K, we denote by $HS(H;K)$ the space of Hilbert–Schmidt operators ϕ from H to K, endowed with the norm:

$$\|\phi\|_{HS(H;K)} = \Big(\sum_{n\in\mathbb{N}} \|\phi e_n\|_K^2\Big)^{\frac{1}{2}},$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H.

Since our focus is the mass-critical and energy-critical cases, we introduce $\mathcal{N}_k(u)$, $k = 0, 1$, by

$$\mathcal{N}_0(u) := |u|^{\frac{4}{d}}u \quad \text{ and } \quad \mathcal{N}_1(u) := |u|^{\frac{4}{d-2}}u. \tag{8}$$

Namely, $k = 0$ corresponds to the mass-critical case, while $k = 1$ corresponds to the energy-critical case.

The Strichartz estimates play an important role in our analysis. We say that a pair (q, r) is admissible if $2 \le q, r \le \infty$, $(q, r, d) \ne (2, \infty, 2)$, and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

Then, the following Strichartz estimates are known to hold; see [20,23,32,38].

Lemma 2.1 *Let (q, r) be admissible. Then, we have*

$$\|S(t)\phi\|_{L^q_t L^r_x} \lesssim \|\phi\|_{L^2}.$$

For any admissible pair $(\widetilde{q}, \widetilde{r})$, we also have

$$\bigg\|\int_0^t S(t-t')F(t')dt'\bigg\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\widetilde{q}'}_t L^{\widetilde{r}'}_x}, \tag{9}$$

where $\widetilde{q}'$ and $\widetilde{r}'$ denote the Hölder conjugates. Moreover, if the right-hand side of (9) is finite for some admissible pair $(\widetilde{q}, \widetilde{r})$, then $\int_0^t S(t-t')F(t')dt'$ is continuous (in time) with values in $L^2(\mathbb{R}^d)$.

Next, we provide a precise meaning to the stochastic convolution defined in (5). Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$. Fix an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}^d)$. We define an $L^2(\mathbb{R}^d)$-cylindrical Wiener process W by

$$W(t, x, \omega) := \sum_{n\in\mathbb{N}} \beta_n(t, \omega)e_n(x), \tag{10}$$

where $\{\beta_n\}_{n\in\mathbb{N}}$ is a family of mutually independent complex-valued Brownian motions associated with the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. Here, the complex-valued Brownian motion means that $\mathrm{Re}\beta_n(t)$ and $\mathrm{Im}\beta_n(t)$ are independent (real-valued) Brownian motions. Then, the space-time white noise ξ is given by a distributional derivative (in time) of W and thus we can express the stochastic convolution Ψ as

$$\Psi(t) = -i \sum_{n\in\mathbb{N}} \int_0^t S(t-t')\phi e_n \, d\beta_n(t'), \tag{11}$$

where each summand is a classical Wiener integral (with respect to the integrator $d\beta_n$); see [27]. Then, we have the following lemma on the regularity properties of the stochastic convolution. See, for example, Proposition 5.9 in [12] for Part (i). As for Part (ii), see [30].

Lemma 2.2 *Let $d \geq 1$, $T > 0$, and $s \in \mathbb{R}$. Suppose that $\phi \in HS(L^2; H^s)$.*

(i) *We have $\Psi \in C([0, T]; H^s(\mathbb{R}^d))$ almost surely. Moreover, for any finite $p \geq 1$, there exists $C = C(T, p) > 0$ such that*

$$\mathbb{E}\bigg[\sup_{0 \leq t \leq T} \|\Psi(t)\|_{H^s}^p\bigg] \leq C\|\phi\|_{HS(L^2;H^s)}^p.$$

(ii) *Given $1 \leq q < \infty$ and finite $r \geq 2$ such that $r \leq \frac{2d}{d-2}$ when $d \geq 3$, we have $\Psi \in L^q([0, T]; W^{s,r}(\mathbb{R}^d))$ almost surely. Moreover, for any finite $p \geq 1$, there exists $C = C(T, p) > 0$ such that*

$$\mathbb{E}\Big[\|\Psi\|_{L^q([0,T];W^{s,r}(\mathbb{R}^d))}^p\Big] \leq C\|\phi\|_{HS(L^2;H^s)}^p.$$

By the Strichartz estimates (Lemma 2.1) and Lemma 2.2 on the stochastic convolution, one can easily prove the following local well-posedness (see Lemma 2.3 below) of the mass-critical and energy-critical SNLS (1) by essentially following the argument in [14], namely, by studying the Duhamel formulation for $v = u - \Psi$:

$$v(t) = S(t)u_0 - i\int_0^t S(t-t')\mathcal{N}(v + \Psi)(t')dt'.$$

See also Lemmas 3.1 and 4.1 below. In the mass-critical case, the admissible pair $q = r = \frac{2(d+2)}{d}$ plays an important role. In the energy-critical case, we use the following admissible pair

$$(q_d, r_d) := \left(\frac{2d}{d-2}, \frac{2d^2}{d^2 - 2d + 4}\right) \tag{12}$$

for $d \geq 3$.

Lemma 2.3 (i) *(Mass-critical case). Let $d \geq 1$, $p = 1 + \frac{4}{d}$, and $\phi \in HS(L^2; L^2)$. Then, given any $u_0 \in L^2(\mathbb{R}^d)$, there exists an almost surely positive stopping time $T = T_\omega(u_0)$ and a unique local-in-time solution $u \in C([0, T]; L^2(\mathbb{R}^d))$ to the mass-critical SNLS (1). Furthermore, the following blowup alternative holds; let $T^* = T^*_\omega(u_0)$ be the forward maximal time of existence. Then, either*

$$T^* = \infty \qquad or \qquad \lim_{T \nearrow T^*} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,T)\times\mathbb{R}^d)} = \infty.$$

(ii) *(Energy-critical case). Let $3 \leq d \leq 6$, $p = 1 + \frac{4}{d-2}$, and $\phi \in HS(L^2; H^1)$. Then, given any $u_0 \in H^1(\mathbb{R}^d)$, there exists an almost surely positive stopping time $T = T_\omega(u_0)$ and a unique local-in-time solution $u \in C([0, T]; H^1(\mathbb{R}^d))$ to the* *energy-critical SNLS* (1). *Furthermore, the following blowup alternative holds; let* $T^* = T^*_\omega(u_0)$ *be the forward maximal time of existence. Then, either*

$$T^* = \infty \qquad or \qquad \lim_{T \nearrow T^*} \|u\|_{L^{q_d}([0,T);W^{1,r_d}(\mathbb{R}^d))} = \infty.$$

We note that the mapping: $(u_0, \Psi) \mapsto v$ is continuous. See Proposition 3.5 in [14]. In the energy-critical case, the local-in-time well-posedness also holds for $d > 6$ (see Remark 4.2 below). As mentioned earlier, the perturbation argument requires the Lipschitz continuity of $\nabla\mathcal{N}$ and hence we need to assume $d \le 6$ in the following.

Lastly, we state the a priori bounds on the mass and energy of solutions constructed in Lemma 2.3.

Lemma 2.4 (i) *(Mass-critical case). Assume the hypotheses in Lemma* 2.3(i). *Then, given* $T_0 > 0$*, there exists* $C_1 = C_1(M(u_0), T_0, \|\phi\|_{HS(L^2;L^2)}) > 0$ *such that for any stopping time* T *with* $0 < T < \min(T^*, T_0)$ *almost surely, we have*

$$\mathbb{E}\bigg[\sup_{0\le t\le T} M(u(t))\bigg] \le C_1, \tag{13}$$

where u *is the solution to the mass-critical SNLS* (1) *with* $u|_{t=0} = u_0$ *and* $T^* = T^*_\omega(u_0)$ *is the forward maximal time of existence.*

(ii) *(Energy-critical case). Assume the hypotheses in Lemma* 2.3(ii). *Then, given* $T_0 > 0$*, there exists* $C_2 = C_2(M(u_0), E(u_0), T_0, \|\phi\|_{HS(L^2;H^1)}) > 0$ *such that for any stopping time* T *with* $0 < T < \min(T^*, T_0)$ *almost surely, we have*

$$\mathbb{E}\bigg[\sup_{0\le t\le T} M(u(t))\bigg] + \mathbb{E}\bigg[\sup_{0\le t\le T} E(u(t))\bigg] \le C_2,$$

where u *is the solution to the defocusing energy-critical SNLS* (1) *with* $u|_{t=0} = u_0$ *and* $T^* = T^*_\omega(u_0)$ *is the forward maximal time of existence.*

For Part (ii), we need to assume that the equation is defocusing. These a priori bounds follow from Ito's lemma and the Burkholder–Davis–Gundy inequality. In order to justify an application of Ito's lemma, one needs to go through a certain approximation argument. See, for example, Proposition 3.2 in [14]. In our mass-critical and energy-critical settings, however, such an approximation argument is more involved and hence we present a sketch of the argument in "Appendix A".

3 Mass-critical case

In this section, we prove global well-posedness of the defocusing mass-critical SNLS (1) (Theorem 1.1(i)). In Sect. 3.1, we first study the following defocusing mass-critical NLS with a deterministic perturbation:

$$i\partial_t v + \Delta v = \mathcal{N}_0(v + f), \tag{14}$$

where $\mathcal{N}_0$ is as in (8) and f is a given deterministic function, satisfying certain regularity conditions. By applying the perturbation lemma, we prove global existence for (14), assuming an a priori L^2-bound of a solution v to (14). See Proposition 3.3. In Sect. 3.2, we then present the proof of Theorem 1.1(i) by writing (1) in the form (14) (with $f = \Psi$) and verifying the hypotheses in Proposition 3.3.

3.1 Mass-critical NLS with a perturbation

By the standard Strichartz theory, we have the following local well-posedness of the perturbed NLS (14).

Lemma 3.1 *There exists small $\eta_0 > 0$ such that if*

$$\|S(t-t_0)v_0\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le \eta$$

for some $\eta \le \eta_0$ and some time interval $I = [t_0, t_1] \subset \mathbb{R}$, then there exists a unique solution $v \in C(I; L^2(\mathbb{R}^d)) \cap L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbb{R}^d)$ to (14) with $v(t_0) = v_0 \in L^2(\mathbb{R}^d)$. Moreover, we have

$$\|v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le 2\eta.$$

Proof We show that the map Γ defined by

$$\Gamma v(t) := S(t-t_0)v_0 - i\int_{t_0}^{t} S(t-t')\mathcal{N}_0(v+f)(t')dt'$$

is a contraction on the ball $B_{2\eta} \subset L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbb{R}^d)$ of radius $2\eta > 0$ centered at the origin, provided that $\eta > 0$ is sufficiently small. Noting that the Hölder conjugate of $\frac{2(d+2)}{d}$ is $\frac{2(d+2)}{d+4} = \frac{2(d+2)}{d}/(1+\frac{4}{d})$, it follows from Lemma 2.1 that there exists small $\eta_0 > 0$ such that

$$\begin{aligned}
\|\Gamma v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} &\le \|S(t-t_0)v_0\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} + \|\Gamma v - S(t-t_0)v_0\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \\
&\le \eta + C\Big(\|v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)}\Big)^{1+\frac{4}{d}} \\
&\le \eta + C\eta^{1+\frac{4}{d}} \le 2\eta
\end{aligned}$$

and

$$\|\Gamma v_1 - \Gamma v_2\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le \frac{1}{2}\|v_1 - v_2\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)}$$

for any $v, v_1, v_2 \in B_{2\eta}$ and $0 < \eta \le \eta_0$. Hence, Γ is a contraction on $B_{2\eta}$. Furthermore, we have

$$\begin{aligned}\|v\|_{L^\infty(I;L^2(\mathbb{R}^d))} &\le \|S(t-t_0)v_0\|_{L^\infty(I;L^2(\mathbb{R}^d))} \\ &\quad + C\Big(\|v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)}\Big)^{1+\frac{4}{d}} \\ &\le \|v_0\|_{L^2} + C\eta^{1+\frac{4}{d}} < \infty\end{aligned}$$

for any $v \in B_{2\eta}$. This shows that $v \in C(I; L^2(\mathbb{R}^d))$. □

Next, we recall the long-time stability result in the mass-critical setting. See [35] for the proof.

Lemma 3.2 (Mass-critical perturbation lemma) *Let I be a compact interval. Suppose that $v \in C(I; L^2(\mathbb{R}^d))$ satisfies the following perturbed NLS:*

$$i\partial_t v + \Delta v = |v|^{\frac{4}{d}} v + e, \tag{15}$$

satisfying

$$\|v\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le R$$

for some $R \ge 1$. Then, there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that if we have

$$\|u_0 - v(t_0)\|_{L^2(\mathbb{R}^d)} + \|e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I\times\mathbb{R}^d)} \le \varepsilon \tag{16}$$

for some $u_0 \in L^2(\mathbb{R}^d)$, some $t_0 \in I$, and some $\varepsilon < \varepsilon_0$, then there exists a solution $u \in C(I; L^2(\mathbb{R}^d))$ to the defocusing mass-critical NLS:

$$i\partial_t u + \Delta u = |u|^{\frac{4}{d}} u \tag{17}$$

with $u(t_0) = u_0$ such that

$$\begin{aligned}\|u\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} &\le C_1(R), \\ \|u-v\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|u-v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} &\le C_1(R)\varepsilon,\end{aligned}$$

where $C_1(R)$ is a non-decreasing function of R.

In the remaining part of this subsection, we consider long time existence of solutions to the perturbed NLS (14) under several assumptions. Given $T > 0$, we assume that there exist $C, \theta > 0$ such that

$$\|f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I\times\mathbb{R}^d)} \le C|I|^\theta \tag{18}$$

for any interval $I \subset [0, T]$. Then, Lemma 3.1 guarantees existence of a solution to the perturbed NLS (14), at least for a short time. The following proposition establishes long time existence under some hypotheses.

Proposition 3.3 *Given $T > 0$, assume the following conditions (i)–(ii):*

(i) $f \in L_{t,x}^{\frac{2(d+2)}{d}}([0, T] \times \mathbb{R}^d)$ *satisfies* (18),
(ii) *Given a solution v to* (14)*, the following a priori L^2-bound holds:*

$$\|v\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} \leq R \tag{19}$$

for some $R \geq 1$.

Then, there exists $\tau = \tau(R, \theta) > 0$ such that, given any $t_0 \in [0, T)$, a unique solution v to (14) *exists on $[t_0, t_0 + \tau] \cap [0, T]$. In particular, the condition (ii) guarantees existence of a unique solution v to the perturbed NLS* (14) *on the entire interval* $[0, T]$.

Proof By setting $e = \mathcal{N}_0(v + f) - \mathcal{N}_0(v)$, Eq. (14) reduces to (15). In the following, we iteratively apply Lemma 3.2 on short intervals and show that there exists $\tau = \tau(R, \theta) > 0$ such that (15) is well-posed on $[t_0, t_0 + \tau] \cap [0, T]$ for any $t_0 \in [0, T)$.

Let w be the global solution to the defocusing mass-critical NLS (17) with $w(t_0) = v(t_0) = v_0$. By the assumption (19), we have $\|w(t_0)\|_{L^2} \leq R$. Then, by the space-time bound (3), we have

$$\|w\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbb{R}\times\mathbb{R}^d)} \leq C(R) < \infty.$$

Given small $\eta > 0$ (to be chosen later), we divide the interval $[t_0, T]$ into $J = J(R, \eta) \sim \big(C(R)/\eta\big)^{\frac{2(d+2)}{d}}$ many subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|w\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)} \leq \eta. \tag{20}$$

We point out that η will be chosen as an absolute constant and hence dependence of other constants on η is not essential in the following. Fix $\tau > 0$ (to be chosen later in terms of R and θ) and write $[t_0, t_0+\tau] = \bigcup_{j=0}^{J'} \big([t_0, t_0+\tau]\cap I_j\big)$ for some $J' \leq J-1$, where $[t_0, t_0 + \tau] \cap I_j \neq \emptyset$ for $0 \leq j \leq J'$ and $[t_0, t_0 + \tau] \cap I_j = \emptyset$ for $j > J'$.

Since the nonlinear evolution w is small on each I_j, it follows that the linear evolution $S(t - t_j)w(t_j)$ is also small on each I_j. Indeed, from the Duhamel formula, we have

$$S(t - t_j)w(t_j) = w(t) - i \int_{t_j}^{t} S(t - t')\mathcal{N}_0(w)(t')dt'.$$

Then, by Lemma 2.1 and (20), we have

$$\begin{aligned}\|S(t-t_j)w(t_j)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)} &\le \|w\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)} + C\|w\|^{1+\frac{4}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j\times\mathbb{R}^d)}\\ &\le \eta + C\eta^{1+\frac{4}{d}}\\ &\le 2\eta \end{aligned} \tag{21}$$

for all $j=0,\ldots,J-1$, provided that $\eta>0$ is sufficiently small.

Now, we estimate v on the first interval I_0. By $v(t_0)=w(t_0)$ and (21), we have

$$\|S(t-t_0)v(t_0)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0\times\mathbb{R}^d)} = \|S(t-t_0)w(t_0)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0\times\mathbb{R}^d)} \le 2\eta.$$

Let $\eta_0>0$ be as in Lemma 3.1. Then, by the local theory (Lemma 3.1), we have

$$\|v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0\times\mathbb{R}^d)} \le 6\eta,$$

as long as $3\eta<\eta_0$ and $\tau=\tau(\eta,\theta)=\tau(\theta)>0$ is sufficiently small so that

$$\|f\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0,t_0+\tau))} \le C\tau^\theta \le \eta. \tag{22}$$

Next, we estimate the error term. By Lemma 2.1 and (18), we have

$$\begin{aligned}\|e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_0\times\mathbb{R}^d)} &\le C\bigg(\|v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0\times\mathbb{R}^d)} + \|f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0\times\mathbb{R}^d)}\bigg)^{\frac{4}{d}}\|f\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0\times\mathbb{R}^d)}\\ &\le C\Big(\eta+\tau^\theta\Big)^{\frac{4}{d}}\tau^\theta\\ &\le C\tau^\theta \end{aligned} \tag{23}$$

for any small $\eta,\tau>0$. Given $\varepsilon>0$, we can choose $\tau=\tau(\varepsilon,\theta)>0$ sufficiently small so that

$$\|e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_0\times\mathbb{R}^d)} \le \varepsilon.$$

In particular, for $\varepsilon<\varepsilon_0$ with $\varepsilon_0=\varepsilon_0(R)>0$ dictated by Lemma 3.2, the condition (16) is satisfied on I_0. Hence, by the perturbation lemma (Lemma 3.2), we obtain

$$\|w-v\|_{L^\infty(I_0;L^2(\mathbb{R}^d))} + \|w-v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0\times\mathbb{R}^d)} \le C_1(R)\varepsilon.$$

In particular, we have

$$\|w(t_1)-v(t_1)\|_{L^2(\mathbb{R}^d)} \le C_1(R)\varepsilon. \tag{24}$$

We now move onto the second interval I_1. By (21) and Lemma 2.1 with (24), we have

$$\begin{aligned}
&\|S(t-t_1)v(t_1)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_1\times\mathbb{R}^d)}\\
&\quad\le \|S(t-t_1)w(t_1)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_1\times\mathbb{R}^d)} + \|S(t-t_1)(w(t_1)-v(t_1))\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_1\times\mathbb{R}^d)}\\
&\quad\le 2\eta + C_0\cdot C_1(R)\varepsilon \le 3\eta
\end{aligned} \tag{25}$$

by choosing $\varepsilon = \varepsilon(R,\eta) = \varepsilon(R) > 0$ sufficiently small.

Proceeding as before, it follows from Lemma 3.1 with (25) that

$$\|v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_1\times\mathbb{R}^d)} \le 8\eta,$$

as long as $4\eta \le \eta_0$ and $\tau > 0$ is sufficiently small so that (22) is satisfied. By repeating the computation in (23) with (18), we have

$$\|e\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I_1\times\mathbb{R}^d)} \le C\tau^\theta \le \varepsilon$$

by choosing $\tau = \tau(\varepsilon,\theta) > 0$ sufficiently small. Hence, by the perturbation lemma (Lemma 3.2) applied to the second interval I_1, we obtain

$$\|w-v\|_{L^\infty(I_1;L^2(\mathbb{R}^d))} + \|w-v\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_1\times\mathbb{R}^d)} \le C_1(R)(C_1(R)+1)\varepsilon.$$

provided that $\tau = \tau(\varepsilon,\theta) > 0$ is chosen sufficiently small and that $(C_1(R)+1)\varepsilon < \varepsilon_0$. In particular, we have

$$\|w(t_2)-v(t_2)\|_{L^2(\mathbb{R}^d)} \le C_1(R)(C_1(R)+1)\varepsilon =: C_2(R)\varepsilon.$$

For $j \ge 2$, define $C_j(R)$ recursively by setting

$$C_j(R) = C_1(R)(C_{j-1}(R)+1).$$

Then, proceeding inductively, we obtain

$$\|w(t_j)-v(t_j)\|_{L^2(\mathbb{R}^d)} \le C_j(R)\varepsilon,$$

for all $0 \le j \le J'$, as long as $\varepsilon = \varepsilon(R,\eta,J) > 0$ is sufficiently small such that

- $C_0\cdot C_j(R)\varepsilon \le \eta$ (here, C_0 is the constant from the Strichartz estimate in (25)),
- $(C_j(R)+1)\varepsilon < \varepsilon_0$,

for $j = 1,\dots,J'$. Recalling that $J'+1 \le J = J(R,\eta)$, we see that this can be achieved by choosing small $\eta > 0$, $\varepsilon = \varepsilon(R,\eta) = \varepsilon(R) > 0$, and $\tau = \tau(\varepsilon,\theta) = \tau(R,\theta) > 0$ sufficiently small. This guarantees existence of a (unique) solution v to (14) on $[t_0, t_0+\tau]$. Lastly, noting that $\tau > 0$ is independent of $t_0 \in [0,T)$, we conclude existence of the solution v to (14) on the entire interval $[0,T]$. $\square$

3.2 Proof of Theorem 1.1(i)

We are now ready to present a proof of Theorem 1.1(i). Given a local-in-time solution u to (1), let $v = u - \Psi$. Then, v satisfies

$$\begin{cases} i\partial_t v + \Delta v = \mathcal{N}_0(v + \Psi) \\ v|_{t=0} = u_0. \end{cases} \tag{26}$$

Theorem 1.1(i) follows from applying Proposition 3.3 to (26) with $f = \Psi$, once we verify the hypotheses (i) and (ii).

Fix $T > 0$. From Lemma 2.4 and Markov's inequality, we have the following almost sure a priori bound:

$$\sup_{0 \le t \le T} M(u(t)) \le C\big(\omega, T, M(u_0), \|\phi\|_{HS(L^2;L^2)}\big) < \infty \tag{27}$$

for a solution u to (1) with $p = 1 + \frac{4}{d}$. Then, from (27) and Lemma 2.2(i), we obtain

$$\begin{aligned} \sup_{0 \le t \le T} M(v(t)) = \sup_{0 \le t \le T} M(u(t) - \Psi(t)) &\le \sup_{0 \le t \le T} M(u(t)) + \sup_{0 \le t \le T} M(\Psi(t)) \\ &\le C\big(\omega, T, M(u_0), \|\phi\|_{HS(L^2;L^2)}\big) < \infty \end{aligned}$$

almost surely. This shows that the hypothesis (ii) in Proposition 3.3 holds almost surely for some almost surely finite $R = R(\omega) \ge 1$. The hypothesis (i) in Proposition 3.3 easily follows from Hölder's inequality in time, Markov's inequality, and Lemma 2.2(ii). More precisely, by fixing finite $q > \frac{2(d+2)}{d}$ and noting $\frac{2(d+2)}{d} \le \frac{2d}{d-2}$ for $d \ge 3$, Lemma 2.2(ii) yields

$$\mathbb{E}\Big[\|\Psi\|_{L^q([0,T];L^{\frac{2(d+2)}{d}}(\mathbb{R}^d))}\Big] \le C\|\phi\|_{HS(L^2;L^2)}.$$

Then, Markov's inequality yields

$$\|\Psi\|_{L^q([0,T];L^{\frac{2(d+2)}{d}}(\mathbb{R}^d))} \le C\big(\omega, \|\phi\|_{HS(L^2;L^2)}\big) < \infty, \tag{28}$$

which in turn implies $\Psi \in L_{t,x}^{\frac{2(d+2)}{d}}([0,T] \times \mathbb{R}^d)$ almost surely. Moreover, it follows from (28) and Hölder's inequality in time that

$$\|\Psi\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbb{R}^d)} \le |I|^\theta \|\Psi\|_{L^q(I;L^{\frac{2(d+2)}{d}}(\mathbb{R}^d))} \le C\big(\omega, \|\phi\|_{HS(L^2;L^2)}\big)|I|^\theta$$

for any interval $I \subset [0,T]$, where $\theta = \frac{d}{2(d+2)} - \frac{1}{q} > 0$. This verifies (18).

Hence, by applying Proposition 3.3, we can construct a solution v to (26) on $[0,T]$. Since the choice of $T > 0$ was arbitrary, this proves Theorem 1.1(i).

4 Energy-critical case

In this section, we prove global well-posedness of the defocusing energy-critical SNLS (1) (Theorem 1.1(ii)). The idea is to follow the argument for the mass-critical case presented in Sect. 3. Namely, we study the following defocusing energy-critical NLS with a deterministic perturbation:

$$i\partial_t v + \Delta v = \mathcal{N}_1(v+f), \tag{29}$$

where $\mathcal{N}_1$ is as in (8) and f is a given deterministic function, satisfying certain regularity conditions.

Let q_d and r_d be as in (12) and set $\rho_d := \frac{2d^2}{(d-2)^2}$ for $d \geq 3$. A direct calculation shows that

$$\frac{d+2}{d-2}\frac{1}{q_d} = \frac{1}{q_d'}, \quad \frac{1}{r_d'} = \frac{1}{r_d} + \frac{4}{d-2}\frac{1}{\rho_d}, \quad \text{and} \quad W^{1,r_d}(\mathbb{R}^d) \hookrightarrow L^{\rho_d}(\mathbb{R}^d). \tag{30}$$

4.1 Energy-critical NLS with a perturbation

We first go over the local theory for the perturbed NLS (29) in the energy-critical case.

Lemma 4.1 *Let $3 \leq d \leq 6$. Then, there exists small $\eta_0 = \eta_0 > 0$ such that if*

$$\|S(t-t_0)v_0\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} + \|f\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \leq \eta \tag{31}$$

for some $\eta \leq \eta_0$ and some time interval $I = [t_0, t_1] \subset \mathbb{R}$, then there exists a unique solution $v \in C(I; H^1(\mathbb{R}^d)) \cap L^{q_d}(I; W^{1,r_d}(\mathbb{R}^d))$ to (29) with $v(t_0) = v_0 \in H^1(\mathbb{R}^d)$. Moreover, we have

$$\|v\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \leq 2\eta.$$

Proof We show that the map Γ defined by

$$\Gamma v(t) := S(t-t_0)v_0 - i\int_{t_0}^{t} S(t-t')\mathcal{N}_1(v+f)(t')dt'$$

is a contraction on $B_{2\eta} \subset L^{q_d}(I; W^{1,r_d}(\mathbb{R}^d))$ of radius $2\eta > 0$ centered at the origin, provided that $\eta > 0$ is sufficiently small. It follows from Lemma 2.1 and (30) with (31) that there exists small $\eta_0 > 0$ such that

$$\begin{aligned}\|\Gamma v\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} &\leq \|S(t-t_0)v_0\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} + C\|\mathcal{N}_1(v+f)\|_{L^{q_d'}(I;W^{1,r_d'}(\mathbb{R}^d))}\\ &\leq \eta + C\Big(\|v\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} + \|f\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))}\Big)^{1+\frac{4}{d-2}}\\ &\leq \eta + C\eta^{1+\frac{4}{d-2}} \leq 2\eta\end{aligned}$$

for $v \in B_{2\eta}$ and $0 < \eta \le \eta_0$. Recall that $\nabla \mathcal{N}_1$ is Lipschitz continuous when $3 \le d \le 6$ and we have

$$\begin{aligned} \nabla N_1(u_1) - \nabla \mathcal{N}_1(u_2) = O\big(|u_1|^{\frac{4}{d-2}} + |u_1|^{\frac{4}{d-2}}\big)\nabla(u_1 - u_2) \\ + O\big((|u_1|^{\frac{6-d}{d-2}} + |u_1|^{\frac{6-d}{d-2}})|u_1 - u_2|\big)\nabla u_2. \end{aligned} \tag{32}$$

See, for example, Case 4 in the proof of Proposition 4.1 in [29]. Then, proceeding as above with (32), we have

$$\begin{aligned} &\|\Gamma v_1 - \Gamma v_2\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \\ &\quad \le C\|\mathcal{N}_1(v_1+f) - \mathcal{N}_1(v_2+f)\|_{L^{q'_d}(I;W^{1,r'_d}(\mathbb{R}^d))} \\ &\quad \le C\bigg(\|v_1\|_{L^{q_d}(I;L^{\rho_d}(\mathbb{R}^d))} + \|v_2\|_{L^{q_d}(I;L^{\rho_d}(\mathbb{R}^d))} + \|f\|_{L^{q_d}(I;L^{\rho_d}(\mathbb{R}^d))}\bigg)^{\frac{4}{d-2}} \\ &\qquad \times \|v_1 - v_2\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \\ &\qquad + C\bigg(\|v_1\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} + \|v_2\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} + \|f\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))}\bigg) \\ &\qquad \times \bigg(\|v_1\|_{L^{q_d}(I;L^{\rho_d}(\mathbb{R}^d))} + \|v_2\|_{L^{q_d}(I_0;L^{\rho_d}(\mathbb{R}^d))} + \|f\|_{L^{q_d}(I;L^{\rho_d}(\mathbb{R}^d))}\bigg)^{\frac{6-d}{d-2}} \\ &\qquad \times \|v_1 - v_2\|_{L^{q_d}(I;L^{\rho_d}(\mathbb{R}^d))} \\ &\quad \le C\eta^{\frac{4}{d-2}}\|v_1 - v_2\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \\ &\quad \le \frac{1}{2}\|v_1 - v_2\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \end{aligned} \tag{33}$$

for $v_1, v_2 \in B_{2\eta}$ and $0 < \eta \le \eta_0$. Hence, Γ is a contraction on $B_{2\eta}$. Furthermore, we have

$$\begin{aligned} \|v\|_{L^\infty(I;H^1(\mathbb{R}^d))} &\le \|S(t-t_0)v_0\|_{L^\infty(I;H^1(\mathbb{R}^d))} + C\|\mathcal{N}_1(v+f)\|_{L^{q'_d}(I;W^{1,r'_d}(\mathbb{R}^d))} \\ &\le \|v_0\|_{H^1} + C\eta^{1+\frac{4}{d-2}} < \infty \end{aligned}$$

for $v \in B_{2\eta}$. This shows that $v \in C(I; H^1(\mathbb{R}^d))$. $\square$

Remark 4.2 The restriction $d \le 6$ appears in (32) and (33), where we used the Lipschitz continuity of $\nabla \mathcal{N}_1$. Following the argument in [6], we can remove this restriction and construct a solution by carrying out a contraction argument on $B_{2\eta} \subset L^{q_d}(I; W^{1,r_d}(\mathbb{R}^d))$ equipped with the distance

$$d(v_1, v_2) = \|v_1 - v_2\|_{L^{q_d}(I;L^{r_d}(\mathbb{R}^d))}.$$

Indeed, a slight modification of the computation in (33) shows $d(\Gamma v_1, \Gamma v_2) \le \frac{1}{2}d(v_1, v_2)$ for any $v_1, v_2 \in B_{2\eta}$.

Next, we state the long-time stability result in the energy-critical setting. See [11, 25,33,35]. The following lemma is stated in terms of non-homogeneous spaces, the proof follows closely to that in the mass-critical case.

Lemma 4.3 (Energy-critical perturbation lemma) *Let $3 \leq d \leq 6$ and I be a compact interval. Suppose that $v \in C(I; H^1(\mathbb{R}^d))$ satisfies the following perturbed NLS:*

$$i\partial_t v + \Delta v = |v|^{\frac{4}{d-2}} v + e,$$

satisfying

$$\|v\|_{L^\infty(I;H^1(\mathbb{R}^d))} + \|v\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \leq R$$

for some $R \geq 1$. Then, there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that if we have

$$\|u_0 - v(t_0)\|_{H^1(\mathbb{R}^d)} + \|e\|_{L^{q_d'}(I;W^{1,r_d'}(\mathbb{R}^d))} \leq \varepsilon$$

for some $u_0 \in H^1(\mathbb{R}^d)$, some $t_0 \in I$, and some $\varepsilon < \varepsilon_0$, then there exists a solution $u \in C(I; H^1(\mathbb{R}^d))$ to the defocusing energy-critical NLS:

$$i\partial_t u + \Delta u = |u|^{\frac{4}{d-2}} u$$

with $u(t_0) = u_0$ such that

$$\|u\|_{L^\infty(I;H^1(\mathbb{R}^d))} + \|u\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \leq C_1(R),$$
$$\|u - v\|_{L^\infty(I;H^1(\mathbb{R}^d))} + \|u - v\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \leq C_1(R)\varepsilon,$$

where $C_1(R)$ is a non-decreasing function of R.

With Lemmas 4.1 and 4.3 in hand, we can repeat the argument in Proposition 3.3 and obtain the following proposition. The proof is essentially identical to that of Proposition 3.3 and hence we omit details. We point out that, in applying the perturbation lemma (Lemma 4.3) with $e = \mathcal{N}_1(v + f) - \mathcal{N}_1(v)$, we use (32), which imposes the restriction $d \leq 6$.

Proposition 4.4 *Let $3 \leq d \leq 6$. Given $T > 0$, assume the following conditions (i)–(ii):*

(i) *$f \in L^{q_d}([0, T]; W^{1,r_d}(\mathbb{R}^d))$ and there exist $C, \theta > 0$ such that*

$$\|f\|_{L^{q_d}(I;W^{1,r_d}(\mathbb{R}^d))} \leq C|I|^\theta$$

for any interval $I \subset [0, T]$.

(ii) *Given a solution v to (29), the following a priori H^1-bound holds:*

$$\|v\|_{L^\infty([0,T];H^1(\mathbb{R}^d))} \leq R$$

for some $R \geq 1$.

Then, there exists $\tau = \tau(R, \theta) > 0$ *such that, given any* $t_0 \in [0, T)$*, a unique solution* v *to* (29) *with* $k = 1$ *exists on* $[t_0, t_0 + \tau] \cap [0, T]$*. In particular, the condition (ii) guarantees existence of a unique solution* v *to the perturbed NLS* (29) *on the entire interval* $[0, T]$.

4.2 Proof of Theorem 1.1(ii)

As in Sect. 3.2, Theorem 1.1(ii) follows from applying Proposition 4.4 to (29) with $f = \Psi$, once we verify the hypotheses (i) and (ii).

Fix $T > 0$. As in Sect. 3.2, the hypothesis (i) in Proposition 4.4 can easily be verified from Hölder's inequality in time, Markov's inequality, and Lemma 2.2(ii), once we note that

$$r_d = \frac{2d^2}{d^2 - 2d + 4} \leq \frac{2d}{d-2}.$$

Furthermore, the following almost sure a priori bound follows from Lemma 2.4 and Markov's inequality:

$$\sup_{0 \leq t \leq T} \Big(M(u(t)) + E(u(t)) \Big) \leq C\big(\omega, T, M(u_0), E(u_0), \|\phi\|_{HS(L^2; H^1)}\big) < \infty \tag{34}$$

for a solution u to (1) with $p = 1 + \frac{4}{d-2}$. Then, from (34) and Lemma 2.2(i), we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \|v(t)\|_{H^1} &\leq \sup_{0 \leq t \leq T} \|u(t)\|_{H^1} + \sup_{0 \leq t \leq T} \|\Psi(t)\|_{H^1} \\ &\leq C\big(\omega, T, M(u_0), E(u_0), \|\phi\|_{HS(L^2; H^1)}\big) < \infty \end{aligned}$$

almost surely. This shows that the hypothesis (ii) in Proposition 4.4 holds almost surely for some almost surely finite $R = R(\omega) \geq 1$. This proves Theorem 1.1(ii).

Acknowledgements T.O. was supported by the European Research Council (Grant No. 637995 "ProbDynDispEq"). M.O. was supported by JSPS KAKENHI Grant No. JP16K17624. The authors would like to thank the anonymous referee for helpful comments.

Appendix A: On the application of Ito's lemma

In this appendix, we briefly discuss the derivation of the a priori bounds on the mass and the energy stated in Lemma 2.4. The argument essentially follows from that by de Bouard-Debussche [14] but we indicate certain required modifications.

A.1. Mass-critical case

We first consider the mass-critical case. Given $N \in \mathbb{N}$, let P_N denote a smooth frequency projection onto $\{|\xi| \leq N\}$ and set $\phi_N := P_N \circ \phi$. Then, consider the following truncated SNLS:

$$\begin{cases} i\partial_t u_N + \Delta u_N = \mathcal{N}_0(u_N) + \phi_N \xi \\ u_N|_{t=0} = P_N u_0, \end{cases} \tag{A.1}$$

where $\mathcal{N}_0$ is as in (8). Note that $P_N u_0 \in H^1(\mathbb{R}^d)$ and $\phi_N \in HS(L^2; H^1)$. Therefore, it follows from [14] that (A.1) is globally well-posed for each $N \in \mathbb{N}$. Furthermore, from Proposition 3.2 in [14], we have

$$M(u_N(t)) = M(P_N u_0) + 2\mathrm{Im} \sum_{n \in \mathbb{N}} \int_0^t \int_{\mathbb{R}^d} \overline{u_N(t')} \phi_N e_n dx d\beta_n(t') + 2t \|\phi_N\|^2_{HS(L^2;L^2)} \tag{A.2}$$

for any $t \geq 0$ and, as a consequence of (A.2) and the Burkholder–Davis–Gundy inequality (see, for example, [21, Theorem 3.28 on p. 166]), the a priori bound (13) holds for each u_N, with the constant C_1, independent of $N \in \mathbb{N}$.

Given $T_0 > 0$, let $0 < T < \min(T^*, T_0)$ be a given stopping time as in Lemma 2.4(i) and u be the solution to (1) constructed in Lemma 2.3(i). We now show that the solution u_N to the truncated SNLS (A.1) converges to u almost surely. Then, the a priori bound (13) for u follows from that for u_N mentioned above and the convergence of u_N to u.

In the following, we suppress the spatial domain $\mathbb{R}^d$ for simplicity of the presentation. Given $R > 0$, define a stopping time T_1 by setting

$$T_1 = T_1(R) := \inf \left\{ \tau \geq 0 : \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])} \geq R \right\}$$

and set $T_2 := \min(T, T_1)$. In view of the blowup alternative in Lemma 2.3, we have

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,T])} < \infty$$

almost surely and hence we conclude that $T_2 \nearrow T$ almost surely as $R \to \infty$.

Given small $\eta > 0$ (to be chosen later), we divide the interval $[0, T_2]$ into $J = J(R, \eta)$ many random subintervals $I_j = I_j(\omega) = [t_j, t_{j+1}]$ with $t_0 = 0 < t_1 < \cdots <$

$t_{J-1} < t_J = T_2$ such that

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j)} \sim \eta \tag{A.3}$$

for $j = 0, 1, \dots, J-1$.

Define the truncated stochastic convolution Ψ_N by

$$\Psi_N(t) := -i \sum_{n\in\mathbb{N}} \int_0^t S(t-t')\phi_N e_n d\beta_n(t')$$

and set

$$\begin{aligned} C_N^{(j)}(\omega, u_0, \phi) &= \|u(t_j) - u_N(t_j)\|_{L^2} \\ &\quad + \|\Psi - \Psi_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,T_2])} + \|\Psi - \Psi_N\|_{L^\infty([0,T_2];L^2)} \end{aligned} \tag{A.4}$$

for $j = 0, 1, \dots, J-1$. Then, it follows from the Lebesgue dominated convergence theorem (applied to $(\mathrm{Id} - P_N)u_0$) and Lemma 2.2 that

$$C_N^{(0)}(\omega, u_0, \phi) \longrightarrow 0 \tag{A.5}$$

almost surely as $N \to \infty$.

From the Strichartz estimates (Lemma 2.1), we have

$$\begin{aligned} \|u - u_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])} &\lesssim \|u(0) - u_N(0)\|_{L^2} \\ &\quad + \Big(\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])} + \|u_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])}\Big)^{\frac{4}{d}} \|u - u_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])} \\ &\quad + \|\Psi - \Psi_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])} \end{aligned} \tag{A.6}$$

for any subinterval $[0, \tau] \subset I_0 = [0, t_1]$. Then, from (A.6) with (A.3) and (A.4), we obtain

$$\begin{aligned} \|u - u_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])} &\lesssim C_N^{(0)}(\omega, u_0, \phi) \\ &\quad + \Big(\eta + \|u_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])}\Big)^{\frac{4}{d}} \|u - u_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}([0,\tau])} \end{aligned} \tag{A.7}$$

for any $0 \le \tau \le t_1$. By taking $\eta > 0$ sufficiently small, a standard continuity argument with (A.7) and (A.5) yields

$$\|u_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_0)} \sim \eta, \tag{A.8}$$

uniformly in $N \ge N_0(\omega)$.

Applying Lemma 2.1 once again with (A.3) and (A.8), we then have

$$\begin{aligned}\|u-u_N\|_{L^\infty(I_0;L^2)}+\|u-u_N\|_{L^{\frac{2(d+2)}{d}}_{t,x}(I_0)}&\lesssim \|u(0)-u_N(0)\|_{L^2}+\eta^{\frac{4}{d}}\|u-u_N\|_{L^{\frac{2(d+2)}{d}}_{t,x}(I_0)}\\&\quad+\|\Psi-\Psi_N\|_{L^\infty(I_0;L^2)}+\|\Psi-\Psi_N\|_{L^{\frac{2(d+2)}{d}}_{t,x}(I_0)}\end{aligned}$$

uniformly in $N\geq N_0(\omega)$. Thus, from (A.4) and (A.5), we conclude that

$$\|u-u_N\|_{L^\infty(I_0;L^2)}\lesssim C_N^{(0)}(\omega,u_0,\phi)\longrightarrow 0$$

as $N\to\infty$. In particular, we have

$$\|u(t_1)-u_N(t_1)\|_{L^2}\lesssim C_N^{(0)}(\omega,u_0,\phi)\longrightarrow 0 \tag{A.9}$$

as $N\to\infty$. By repeating the argument above, we have

$$\|u_N\|_{L^{\frac{2(d+2)}{d}}_{t,x}(I_1)}\sim\eta,$$

uniformly in $N\geq N_1(\omega)$. Together with (A.9), this yields

$$\|u-u_N\|_{L^\infty(I_1;L^2)}\lesssim C_N^{(1)}(\omega,u_0,\phi)\longrightarrow 0$$

as $N\to\infty$.

By successively applying the argument above to the interval I_j, $j=0,1,\dots,J-1$, we conclude that

$$\|u-u_N\|_{L^\infty(I_j;L^2)}\lesssim C_N^{(j)}(\omega,u_0,\phi)\longrightarrow 0$$

as $N\to\infty$. Therefore, recalling that $J=J(R,\eta)$ depends only on $R>0$ and an absolute constant $\eta>0$, we obtain

$$\|u-u_N\|_{L^\infty([0,T_2];L^2)}\lesssim\sum_{j=0}^{J-1}C_N^{(j)}(\omega,u_0,\phi)\longrightarrow 0.$$

By the almost sure convergence of u_N to u in $C([0,T_2];L^2(\mathbb{R}^d))$, Fatou's lemma, and the uniform bound (13) for u_N, we then have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq t\leq T_2}M(u(t))\Big]&=\mathbb{E}\Big[\lim_{N\to\infty}\sup_{0\leq t\leq T_2}M(u_N(t))\Big]\\&\leq\lim_{N\to\infty}\mathbb{E}\Big[\sup_{0\leq t\leq T_2}M(u_N(t))\Big]\leq C_1.\end{aligned}$$

Finally, from the almost sure convergence of $T_2 = T_2(R)$ to T, as $R \to \infty$, and Fatou's lemma, we conclude the bound (13) for u. This proves Lemma 2.4(i).

A.2. Energy-critical case

Next, we consider the energy-critical case. In the following, we only discuss the a priori bound on the energy:

$$\mathbb{E}\bigg[\sup_{0\le t\le T} E(u(t))\bigg] \le C_2, \tag{A.10}$$

since the a priori bound on the mass follows in a similar but simpler manner.

Lemma 4.5 *Assume the hypotheses in Lemma* 2.3(ii). *Then, for any stopping time T such that $0 < T < T^*$ almost surely, we have*

$$\begin{aligned} E(u(T)) = E(u_0) &- \operatorname{Im} \sum_{n\in\mathbb{N}} \int_0^T \int_{\mathbb{R}^d} \overline{(\Delta u - \mathcal{N}_1(u))(t')}\phi e_n dx d\beta_n(t') \\ &+ \frac{d}{d-2} \sum_{n\in\mathbb{N}} \int_0^T \int_{\mathbb{R}^d} |u(t')|^{\frac{4}{d-2}} |\phi e_n|^2 dx dt' + T\|\phi\|^2_{HS(L^2;\dot{H}^1)}, \end{aligned} \tag{A.11}$$

where u is the solution to the energy-critical SNLS (1) *with $p = 1 + \frac{4}{d-2}$, $\mathcal{N}_1$ is as in* (8)*, and T^* is the forward maximal time of existence.*

Once we prove Lemma 4.5, the bound (A.10) follows from the Burkholder-Davis-Gundy inequality.

Proof A direct calculation shows that

$$\begin{aligned} E'(u(t))(v) &= \operatorname{Re} \int_{\mathbb{R}^d} \Big(\nabla u(t) \cdot \overline{\nabla v} + \mathcal{N}_1(u)(t)\overline{v}\Big) dx, \\ E''(u(t))(v_1, v_2) &= \operatorname{Re} \int_{\mathbb{R}^d} \nabla v_1 \cdot \overline{\nabla v_2} dx + \operatorname{Re} \int_{\mathbb{R}^d} |u(t)|^{\frac{4}{d-2}} v_1 \overline{v_2} dx \\ &\quad + \frac{4}{d-2} \int_{\mathbb{R}^d} |u(t)|^{\frac{4}{d-2}-2} \operatorname{Re}\big(u(t)\overline{v_1}\big) \operatorname{Re}\big(u(t)\overline{v_2}\big) dx \end{aligned}$$

for $v, v_1, v_2 \in H^1(\mathbb{R}^d)$. Thus, a formal application of Ito's lemma to $E(u(t))$ yields (A.11). It remains to justify the application of Ito's lemma.

As in the proof of Proposition 3.3 in [14], given $N \in \mathbb{N}$, we consider the following truncated problem:

$$\begin{cases} i\partial_t u_N + \Delta u_N = P_N \mathcal{N}_1(u_N) + \phi_N \xi \\ u_N|_{t=0} = P_N u_0, \end{cases} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \tag{A.12}$$

where P_N and ϕ_N are the same as those in Sect. 1. Since the frequency truncation is harmless, the same well-posedness result as in Lemma 2.3 (ii) holds for the truncated SNLS (A.12). Moreover, by considering the corresponding Duhamel formulation for (A.12), we have $u_N = P_{3N}u_N$. We can therefore apply Ito's lemma (see Theorem 4.32 in [12]) to $E(u_N(t))$ and obtain

$$\begin{aligned} E(u_N(t)) &= E(P_N u_0) - \operatorname{Im} \sum_{n\in\mathbb{N}} \int_0^t \int_{\mathbb{R}^d} \overline{(\Delta u_N - \mathcal{N}_1(u_N))(t')} \phi_N e_n dx d\beta_n(t') \\ &\quad + \operatorname{Im} \int_0^t \int_{\mathbb{R}^d} \overline{\Delta u_N(t')} (\mathrm{Id} - P_N)\mathcal{N}_1(u_N)(t') dx dt' \\ &\quad + \frac{d}{d-2} \sum_{n\in\mathbb{N}} \int_0^t \int_{\mathbb{R}^d} |u_N(t')|^{\frac{4}{d-2}} |\phi_N e_n|^2 dx dt' + t\|\phi_N\|^2_{HS(L^2;\dot{H}^1)} \end{aligned} \tag{A.13}$$

for $0 < t < T_N^*$, where T_N^* is the forward maximal time of existence for the solution u_N to (A.12).

Given $R > 0$, define a stopping T_1 by setting

$$T_1 = T_1(R) := \inf\left\{\tau \geq 0 : \|u\|_{L^{q_d}([0,\tau];W^{1,r_d})} \geq R\right\}$$

and set $T_2 := \min(T, T_1)$, where T is the stopping time given in Lemma 2.4(ii) with $0 < T < \min(T^*, T_0)$. In view of the blowup alternative in Lemma 2.3, we have

$$\|u\|_{L^{q_d}([0,T];W^{1,r_d})} < \infty \tag{A.14}$$

almost surely and hence we conclude that $T_2 \nearrow T$ almost surely as $R \to \infty$.

From (30) and (32), we have

$$\begin{aligned} &\|\mathcal{N}_1(u) - P_N\mathcal{N}_1(u_N)\|_{L^{q_d'}(I;W^{1,r_d'})} \\ &\quad \lesssim \|(\mathrm{Id} - P_N)\mathcal{N}_1(u)\|_{L^{q_d'}(I;W^{1,r_d'})} + \|\mathcal{N}_1(u) - \mathcal{N}_1(u_N)\|_{L^{q_d'}(I;W^{1,r_d'})} \\ &\quad \lesssim \|(\mathrm{Id} - P_N)\mathcal{N}_1(u)\|_{L^{q_d'}(I;W^{1,r_d'})} \\ &\quad + \left(\|u\|_{L^{q_d}(I;W^{1,r_d})} + \|u_N\|_{L^{q_d}(I;W^{1,r_d})}\right)^{\frac{4}{d-2}} \|u - u_N\|_{L^{q_d}(I;W^{1,r_d})} \end{aligned} \tag{A.15}$$

for any interval $I \subset [0, T_2]$. It follows from the Lebesgue dominated convergence theorem and (A.14) that the first term on the right-hand side of (A.15) converges to 0 almost surely as $N \to \infty$. Accordingly, proceeding as in Sect. 1, we conclude that u_N converges to u in $C([0, T_2]; H^1(\mathbb{R}^d)) \cap L^{q_d}([0, T_2]; W^{1,r_d}(\mathbb{R}^d))$ almost surely. In particular, there exists an almost surely finite number $N_0(\omega) \in \mathbb{N}$ such that $T_N^* \geq T_2$ for any $N \geq N_0(\omega)$ and, as a result, (A.13) holds for any $0 < t < T_2$ and $N \geq N_0(\omega)$. Moreover, from the definition of $T_2 = T_2(R)$, we may assume

$$\|u_N\|_{L^{q_d}([0,T_2];W^{1,r_d})} \leq R + 1 \tag{A.16}$$

for any $N \geq N_0(\omega)$.

This allows us to conclude that the third term on the right-hand side of (A.13) tends to 0 almost surely as $N \to \infty$. Indeed, by (30), (32), (A.14), (A.16), and the almost sure convergence of u_N to u in $L^{q_d}([0, T_2]; W^{1,r_d}(\mathbb{R}^d))$, we have, for any $0 \leq t \leq T_2$,

$$\begin{aligned}
&\bigg| \int_0^t \int_{\mathbb{R}^d} \overline{\Delta u_N(t')} (\mathrm{Id} - P_N) \mathcal{N}_1(u_N)(t') dx dt' \bigg| \\
&\quad \leq \bigg| \int_0^t \int_{\mathbb{R}^d} (\mathrm{Id} - P_N) \overline{\Delta u(t')} \cdot \mathcal{N}_1(u)(t') dx dt' \bigg| \\
&\qquad + \bigg| \int_0^t \int_{\mathbb{R}^d} \overline{(\Delta u - \Delta u_N)(t')} (\mathrm{Id} - P_N) \mathcal{N}_1(u)(t') dx dt' \bigg| \\
&\qquad + \bigg| \int_0^t \int_{\mathbb{R}^d} \overline{\Delta u_N(t')} (\mathrm{Id} - P_N) \big(\mathcal{N}_1(u) - \mathcal{N}_1(u_N)\big)(t') dx dt' \bigg| \\
&\quad \lesssim \|(\mathrm{Id} - P_N) u\|_{L^{q_d}([0,T_2]; W^{1,r_d})} \|\mathcal{N}_1(u)\|_{L^{q_d'}([0,T_2]; W^{1,r_d'})} \\
&\qquad + \|u - u_N\|_{L^{q_d}([0,T_2]; W^{1,r_d})} \|\mathcal{N}_1(u)\|_{L^{q_d'}([0,T_2]; W^{1,r_d'})} \\
&\qquad + \|u_N\|_{L^{q_d}([0,T_2]; W^{1,r_d})} \|\mathcal{N}_1(u) - \mathcal{N}_1(u_N)\|_{L^{q_d'}([0,T_2]; W^{1,r_d'})} \\
&\quad \lesssim \|(\mathrm{Id} - P_N) u\|_{L^{q_d}([0,T_2]; W^{1,r_d})} \|u\|_{L^{q_d}([0,T_2]; W^{1,r_d})}^{\frac{d+2}{d-2}} \\
&\qquad + \Big(\|u\|_{L^{q_d}([0,T_2]; W^{1,r_d})} + \|u_N\|_{L^{q_d}([0,T_2]; W^{1,r_d})} \Big)^{\frac{d+2}{d-2}} \|u - u_N\|_{L^{q_d}([0,T_2]; W^{1,r_d})} \\
&\qquad \longrightarrow 0
\end{aligned} \tag{A.17}$$

almost surely, as $N \to \infty$.

Let us now consider the second and fourth terms on the right-hand side of (A.13). As for the second term, we first consider the contribution from $\overline{\mathcal{N}_1(u_N)} \phi_N e_n$. By Hölder's inequality with (8) and Sobolev's embedding: $\dot{H}^1(\mathbb{R}^d) \hookrightarrow L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, we have

$$\bigg| \int_{\mathbb{R}^d} \overline{\mathcal{N}_1(u_N)} \phi_N e_n dx \bigg| \leq \|u_N\|_{L^{\frac{2d}{d-2}}}^{\frac{d+2}{d-2}} \|\phi_N e_n\|_{L^{\frac{2d}{d-2}}} \lesssim \|u_N\|_{\dot{H}^1}^{\frac{d+2}{d-2}} \|\phi_N e_n\|_{\dot{H}^1}.$$

Then, by Ito's isometry along with the independence of $\{\beta_n\}_{n \in \mathbb{N}}$, we obtain

$$\begin{aligned}
\mathbb{E}\bigg[\bigg| \sum_{n \in \mathbb{N}} \int_0^t \int_{\mathbb{R}^d} \overline{\mathcal{N}_1(u_N)(t')} \phi_N e_n dx d\beta_n(t') \bigg|^2 \bigg] &\lesssim t \sum_{n \in \mathbb{N}} \|u_N\|_{L^\infty([0,t]; \dot{H}^1)}^{\frac{2(d+2)}{d-2}} \|\phi_N e_n\|_{\dot{H}^1}^2 \\
&\lesssim t \|u_N\|_{L^\infty([0,t]; \dot{H}^1)}^{\frac{2(d+2)}{d-2}} \|\phi_N\|_{HS(L^2; \dot{H}^1)}^2.
\end{aligned} \tag{A.18}$$

By integration by parts (in x) and Ito's isometry, we bound the contribution from $\overline{\Delta u_N} \phi_N e_n$ by

$$\mathbb{E}\bigg[\bigg|\sum_{n\in\mathbb{N}}\int_0^t\int_{\mathbb{R}^d}\overline{\Delta u_N(t')}\phi_N e_n dx d\beta_n(t')\bigg|^2\bigg] \lesssim t\|u_N\|^2_{L^\infty([0,t];\dot{H}^1)}\|\phi_N\|^2_{HS(L^2;\dot{H}^1)}. \tag{A.19}$$

As for the fourth term on the right-hand side of (A.13), it follows from Hölder's and Sobolev's inequalities that

$$\begin{aligned}\sum_{n\in\mathbb{N}}\int_{\mathbb{R}^d}|u_N|^{\frac{4}{d-2}}|\phi_N e_n|^2 dx &\le \|u_N\|^{\frac{4}{d-2}}_{L^{\frac{2d}{d-2}}}\sum_{n\in\mathbb{N}}\|\phi_N e_n\|^2_{L^{\frac{2d}{d-2}}}\\ &\lesssim \|u_N\|^{\frac{4}{d-2}}_{\dot{H}^1}\|\phi_N\|^2_{HS(L^2;\dot{H}^1)}.\end{aligned} \tag{A.20}$$

Since $3 \le d \le 6$, we have $\frac{4}{d-2} \ge 1$, which implies that difference estimates on the contributions from u_N and u for (A.18), (A.19), and (A.20) also hold. Therefore, by in view of (A.17) and (the difference estimates for) (A.18), (A.19), and (A.20), we obtain (A.11) by taking $N \to \infty$ in (A.13) and then $R \to \infty$. This concludes the proof of Lemma 4.5. $\square$

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